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A hierarchy of coupled Korteweg–de Vries equations and the normalisation conditions of the Hilbert–Riemann problem†

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Abstract. We discuss the normalisation conditions of the Hilbert–Riemann problem which is of importance for the construction of the ladder of soliton solutions. For a given spectral problem we provide the prescription for constructing the appropriate normalisation condition. All this is applied to a new hierarchy of coupled Korteweg–de Vries equations to find the soliton solution which displays some peculiar phenomena.

1. Introduction

A few years ago Zakharov and Shabat (1980) reduced the integration of solvable nonlinear equations of interest in mathematical physics to the solution of a matrix Hilbert–Riemann problem. Most of the early literature (Zakharov and Mikhailov 1978, Zakharov and Manakov 1979, Manakov *et al* 1980) on this technique of solving nonlinear equations (sometimes called 'the dressing method' (DM)) considered nonlinear evolution equations (NEEs) whose associated Hilbert–Riemann problem has the so-called canonical normalisation. This is the origin of the belief that the canonical normalisation is generally valid. On the other hand, the calculation of N-soliton solutions by the DM fails unless proper normalisation is applied.

For instance, in the well known case of the KdV equation considered in the framework of the Zakharov–Shabat spectral problem (Ablowitz *et al* 1974) one encounters the necessity for non-canonical normalisation. It turns out that by applying the canonical normalisation it is impossible even to generate a one-soliton solution from the bare solution.

For a given linear problem the proper normalisation condition is provided by the matrix D_1 introduced by Levi *et al* (1982). In principle the normalisation matrix D_1 can depend, through the soliton fields, on the independent variables x and t. Under some reasonable conditions, as will be shown later, the normalisation matrix can be gauge reduced‡ to some constant matrix. But the further reduction to the identity matrix (i.e. to canonical normalisation) is generally impossible. In the above mentioned example of KdV, D_1 is equal to the third Pauli matrix σ_3.

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All these statements prove to be useful for finding soliton solutions in the interesting case of a hierarchy of coupled KdV-like (cKdV) equations (Levi 1981) where the D_1 matrix depends on x and t. This hierarchy has two important features: firstly the odd sub-hierarchy can be reduced to the ordinary KdV hierarchy and secondly its third member resembles the celebrated Hirota–Satsuma system of equations (Hirota and Satsuma 1981, Dodd and Fordy 1982, Wilson 1982).

In § 2 we briefly outline the DM (Zakharov and Shabat 1980) for the construction of soliton solutions and recall (Levi *et al* 1982) the technique necessary to recover the normalisation condition of the Hilbert–Riemann problem. Then we formulate the conditions under which the normalisation matrix can be reduced to a constant matrix. In § 3, using the generalised Lax technique (Bruschi and Ragnisco 1980a), we rederive the cKdV hierarchy, which was previously obtained by the Wronskian technique. By the use of the generalised Lax technique we are also able to obtain the Darboux matrix which cannot be recovered by the Wronskian technique. Finally in § 4 we calculate the soliton solution.

2. Dressing method and the normalisation of the Hilbert–Riemann problem

The matrix Hilbert–Riemann problem may be formulated as follows (Zakharov and Shabat 1980): given a closed contour Γ on the compact complex λ plane and an $N\times N$ matrix function $G(\lambda)$ defined on the contour Γ, we have to determine the following factorisation of the function $G(\lambda)$:

$$G(\lambda)=\psi_1(\lambda)\psi_2(\lambda) \tag{1}$$

where ψ_1 and ψ_2, $N\times N$ matrix functions of λ, originally defined on Γ, are to be extended analytically respectively inside and outside the contour Γ. In this problem all the above matrices can depend on parameters, for instance, x and t. For uniqueness of the solution it is necessary (but not sufficient) to assume the so-called normalisation condition, i.e. $\psi_2(\lambda_0)=\chi$, where λ_0 is some fixed complex number and χ is some fixed $N\times N$ matrix. The canonical normalisation corresponds to the choice $\lambda_0\equiv\infty$ and $\chi=I$.

In the following we shall be exclusively concentrating on the particular Hilbert–Riemann problem defined by requiring that $G=I$ and both functions $\det\psi_1$ and $\det\psi_2$ possess n zeros in their domain of analyticity. From (1) it follows that $\psi_2=\psi_1^{-1}$. One can show that in this case the general solution of the Hilbert–Riemann problem can be cast into the form

$$\begin{aligned}\psi_1(\lambda)&=\psi_1^{(1)}(\lambda)\psi_2^{(2)}(\lambda)\ldots\psi_1^{(n)}(\lambda),\\ \psi_2(\lambda)&=\psi_2^{(n)}(\lambda)\psi_2^{(n-1)}(\lambda)\ldots\psi_2^{(1)}(\lambda),\end{aligned} \tag{2}$$

where the generic $\psi_1^{(j)}$, $\psi_2^{(j)}$ have the structure

$$\psi_1^{(j)}(\lambda)=\chi_j^{-1}\left(I-\frac{\lambda_j-\mu_j}{\lambda-\mu_j}P_j\right),\qquad \psi_2^{(j)}(\lambda)=\left(I+\frac{\lambda_j-\mu_j}{\lambda-\lambda_j}P_j\right)\chi_j, \tag{3}$$

where $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$ are zeros of $\det\psi_1$ and $\det\psi_2$ respectively. Here P_j are λ-independent $N\times N$ projection matrices, $P_j^2=P_j$, while χ_j are non-degenerate normalisation matrices i.e. $\chi=\chi_n\chi_{n-1}\ldots\chi_1$. Solution (2) of the Hilbert–Riemann problem can be used to generate new solutions of NEEs of soliton type.

The generic NEE equation can be written in the form†

$$U_t - V_x + [U, V] = 0 \tag{4}$$

which is the compatibility condition for the linear problems

$$\psi_x = U\psi, \qquad \psi_t = V\psi, \tag{5}$$

where U, V are $N \times N$ matrix functions depending in a preassigned way on a set of soliton fields $q(x, t)$ and on the complex parameter λ; ψ may be understood either as a complex $N \times N$ matrix depending on x, t, λ or as a vector function of the same arguments; $[\cdot, \cdot]$ means the ordinary commutator. Starting from a given solution of (4), (5), i.e. U, V and ψ, a new solution of (5) is defined by $\tilde{\psi} = \psi_2\psi$ where ψ_2 is of the form (2), (3) and $\tilde{\psi}$ is a solution to (5) with U, V replaced by

$$\tilde{U} = \psi_2(-\psi_{1,x} + U\psi_1), \qquad \tilde{V} = \psi_2(-\psi_{1,x} + V\psi_1). \tag{6}$$

Thus $\tilde{U}$, $\tilde{V}$ are a new solution to equation (4). The requirement that $\tilde{U}$, $\tilde{V}$ have the same analytic structure as U, V implies the following equations for the projectors‡ $P_j(x, t)$, $(j = 1, \ldots, n)$:

$$\begin{aligned}
&P_k\chi_k\Big(\prod_{j=1}^{k-1}\psi_1^{(j)}(\lambda_k)\Big)^{-1}[-\partial_x + U(\lambda_k)]\prod_{j=1}^{k-1}\psi_1^{(j)}(\lambda_k)\chi_k^{-1}(I - P_k) = 0,\\
&(I - P_k)\chi_k\Big(\prod_{j=1}^{k-1}\psi_1^{(j)}(\mu_k)\Big)^{-1}[-\partial_x + U(\mu_k)]\prod_{j=1}^{k-1}\psi_1^{(j)}(\mu_k)\chi_k^{-1}P_k = 0,\\
&P_k\chi_k\Big(\prod_{j=1}^{k-1}\psi_1^{(j)}(\lambda_k)\Big)^{-1}[-\partial_t + V(\lambda_k)]\prod_{j=1}^{k-1}\psi_1^{(j)}(\lambda_k)\chi_k^{-1}(I - P_k) = 0,\\
&(I - P_k)\chi_k\Big(\prod_{j=1}^{k-1}\psi_1^{(j)}(\mu_k)\Big)^{-1}[-\partial_t + V(\mu_k)]\prod_{j=1}^{k-1}\psi_1^{(j)}(\mu_k)\chi_k^{-1}P_k = 0.
\end{aligned} \tag{7}$$

Equations (7) determine completely the projection matrices $P_j(x, t)$ in their x and t dependence.

Substitution into (6) of formulae (7) expanded via equations (2), (3) gives the explicit form of the new soliton fields $\tilde{q}(x, t)$ coded in $\tilde{U}$, $\tilde{V}$ in terms of the projection matrices $P_j(x, t)$:

$$\begin{aligned}
\tilde{U} = {}& \sum_{j=1}^{n}\Big(\prod_{k=j+1}^{n}\chi_k\Big)\chi_{j,x}\Big(\prod_{k=j}^{n}\chi_k\Big)^{-1} + \Big(\prod_{j=1}^{n}\chi_j\Big)U(\lambda)\Big(\prod_{j=1}^{n}\chi_j\Big)^{-1}\\
&+\sum_{j=1}^{n}\frac{\lambda_j - \mu_j}{\lambda - \lambda_j}\Big(\prod_{k=j+1}^{n}\psi_1^{(k)}(\lambda_j)\Big)^{-1}P_j\chi_j\Big(\prod_{k=1}^{j-1}\psi_1^{(k)}(\lambda_j)\Big)^{-1}[U(\lambda) - U(\lambda_j)]\\
&\times\prod_{k=1}^{j-1}\psi_1^{(k)}(\lambda_j)\chi_j^{-1}(I - P_j)\prod_{k=j+1}^{n}\psi_1^{(k)}(\lambda_j) - \sum_{j=1}^{n}\frac{\lambda_j - \mu_j}{\lambda - \mu_j}\Big(\prod_{k=j+1}^{n}\psi_1^{(k)}(\mu_j)\Big)^{-1}\\
&\times(I - P_j)\chi_j\Big(\prod_{k=1}^{j-1}\psi_1^{(k)}(\mu_j)\Big)^{-1}\\
&\times[U(\lambda) - U(\mu_j)]\prod_{k=1}^{j-1}\psi_1^{(k)}(\mu_j)\chi_j^{-1}P_j\prod_{k=j+1}^{n}\psi_1^{(k)}(\mu_j).
\end{aligned} \tag{8}$$

† Here and in the following the subscripts x and t mean the corresponding partial derivatives.

‡ The product $\prod_{j=1}^{k-1}\psi_1^{(j)}$ is understood as an ordered product $\psi_1^{(1)}\psi_1^{(2)}\ldots\psi_1^{(k-1)}$ and, as usual, if the superscript of the product is lower than the subscript of the product then the result of the product is the identity matrix I.

A similar formula for $\tilde{V}$ can be obtained from (8) by the following substitutions: U and $\tilde{U}$ go into V and $\tilde{V}$ while the x derivative is replaced by the t derivative. Equation (8) is a generalisation of the formulae given by Levi *et al* (1980) for the case of non-canonical normalisation.

The final computation of $\tilde{U}$ needs also the knowledge of the normalisation matrices χ_j. This can be determined using the equivalence between the Darboux matrix approach and DM as applied for finding soliton solutions (Levi *et al* 1982), namely $\psi_2^{(j)}$ is proportional to $D(\tilde{q}, q, \lambda) = D_0(\tilde{q}, q) + \lambda D_1(\tilde{q}, q)$ with the proportionality factor being a scalar function of λ. As in the asymptotic region $(\lambda \to \infty)$ D behaves like $D_1(\tilde{q}, q)$, the matrix D_1 can be taken as the appropriate normalisation for the Hilbert–Riemann problem. By identifying q with the $(j-1)$-soliton solution q^{j-1} and $\tilde{q}$ with the j-soliton solution q^j we have

$$\chi_j(x, t) = D_1(q^j, q^{j-1}).$$

When the matrix D_1 is non-constant two possibilities can be taken into account: to determine $\tilde{q}$ directly from the formula (8) or to find the appropriate λ-independent gauge transformation necessary to transform the linear problems (5) to the linear problems with constant normalisation. It turns out that the second possibility is computationally simpler whenever the appropriate gauge transformation can be found from the knowledge of the Darboux matrix. To carry out this gauge transformation one should demand that the given D_1 matrix be decomposable in the following way:

$$D_1(\tilde{q}, q) = A^{-1}(\tilde{q})\bar{D}A(q) \tag{9}$$

where A is an $N \times N$ invertible matrix function of its own arguments and $\bar{D}$ is a constant matrix. If the decomposition (9) is fulfilled then by introducing in (5) the new wavefunction $\varphi = A\psi$, the new linear problems

$$\varphi_x = (A_x A^{-1} + AUA^{-1})\varphi, \qquad \varphi_t = (A_t A^{-1} + AVA^{-1})\varphi, \tag{10}$$

have by compatibility the same NEEs as (5), but with constant normalisation matrix $\bar{D}$.

The existence of the decomposition (9) may be proved under the two following general conditions:

(a) $D_1(q, q)$ is a constant matrix;

(b) there exists the adding-two-soliton Darboux matrix, which is a second-order polynomial in λ; this is, in fact, equivalent to the existence of the hierarchy of Bäcklund transformations (Calogero and Degasperis 1977).

From condition (b) and composition properties of the Darboux matrices it turns out that the product $D_1(\tilde{q}, q)D_1(q, q_0)$ is q independent. Therefore

$$D_1(\tilde{q}, q)D_1(q, q_0) = D_1(\tilde{q}, q_0)D_1(q_0, q_0)$$

which implies

$$D_1(\tilde{q}, q) = D_1(\tilde{q}, q_0)D_1(q_0, q_0)D_1^{-1}(q, q_0). \tag{11}$$

3. Coupled KdV equations

3.1. The hierarchy of equations

The hierarchy of coupled Korteweg–de Vries (CKdV) equations together with its Bäcklund transformations (BTs) has been previously derived by Levi (1981) using the Wronskian technique. However, this technique is not able to provide the Darboux matrices corresponding to BTs which, as we saw in § 2, are the necessary tool to obtain the soliton solution via the dressing method. To be able to provide the Darboux matrices we now construct again this hierarchy by using the generalised Lax technique (Bruschi and Ragnisco 1980a). To apply this technique one has to write down the original linear problem

$$\psi_x(x,t)=\begin{pmatrix}0 & -q(x,t)\\ -1 & \lambda-r(x,t)\end{pmatrix}\psi(x,t) \tag{12}$$

in Lax form. Equation (12) cannot be written directly in Lax form since there λ multiplies a singular matrix. Therefore, understanding ψ in formula (12) as a spinor with components φ_1, φ_2, we replace the matrix linear problem (12) by the following scalar linear problem:

$$L\varphi_2(x,t)\equiv\varphi_{2,x}(x,t)+r(x,t)\varphi_2(x,t)+\int_x^\infty \mathrm{d}x'\,q(x',t)\varphi_2(x',t)=\lambda\varphi_2(x,t) \tag{13}$$

completed with the condition

$$\varphi_{1,x}(x,t)=-q(x,t)\varphi_2(x,t). \tag{14}$$

Starting from (13) we look for a NEE written in Lax form, i.e. such that it can be put in the form

$$L_t=[L,M]. \tag{15}$$

Then the problem of finding a hierarchy of NEEs for $q(x,t)$ and $r(x,t)$ reduces to that of finding a sequence of such M operators that the time evolution of $(q(x,t), r(x,t))$ can be cast in the Lax form (15).

According to the algorithm given by Bruschi and Ragnisco (1980a), we assume the existence of an M operator, say M^j, and look for a new operator M, say M^{j+1}, through the ansatz

$$M^{j+1}\varphi_2=LM^j\varphi_2+F^j\varphi_{2,x}+\int_x^\infty \mathrm{d}x'\,G^j\varphi_{2,x'} \tag{16}$$

where F^j and G^j are scalars whose dependence on $(q(x,t), r(x,t))$ is to be determined by the requirement that if $[L,M^j]$ has the form

$$[L,M^j]\varphi_2=V_1^j(x,t)\varphi_2+\int_x^\infty \mathrm{d}x'\,V_2^j(x',t)\varphi_2(x',t)$$

the same holds for $[L,M^{j+1}]$, i.e.

$$[L,M^{j+1}]\varphi_2=V_1^{j+1}(x,t)\varphi_2+\int_x^\infty \mathrm{d}x'\,V_2^{j+1}(x',t)\varphi_2(x',t).$$

These conditions determine F^j and G^j:

$$F^j=\int_x^\infty dx'\, V_1^j(x',t)-F^0, \qquad G^j=-\int_x^\infty dx'\, V_2^j(x',t)$$

(where F^0 is an arbitrary constant) and provide us with the recursion operator which connects $V_1^{j+1}(x,t)$ with $V_1^j(x,t)$ and $V_2^{j+1}(x,t)$ with $V_2^j(x,t)$, i.e.

$$\begin{pmatrix} V_1^{j+1}(x,t) \\ V_2^{j+1}(x,t)\end{pmatrix}=\mathscr{L}\begin{pmatrix} V_1^{j}(x,t) \\ V_2^{j}(x,t)\end{pmatrix}+\begin{pmatrix} V_1^{0}(x,t) \\ V_2^{0}(x,t)\end{pmatrix}$$

where

$$\mathscr{L}\begin{pmatrix} V_1^{j}(x,t) \\ V_2^{j}(x,t)\end{pmatrix}=\begin{pmatrix} V_{1,x}^j(x,t)+r(x,t)V_1^j(x,t)-2V_2^j(x,t)-r_x(x,t)\int_x^\infty dx'\, V_1^j(x',t) \\ -V_{2,x}^j(x,t)+r(x,t)V_2^j(x,t)+2q(x,t)V_1^j(x,t)-q_x(x,t)\int_x^\infty dx'\, V_1^j(x',t)\end{pmatrix},$$

$$\begin{pmatrix} V_1^0 \\ V_2^0\end{pmatrix}=F^0\begin{pmatrix} r_x(x,t) \\ q_x(x,t)\end{pmatrix}.$$

So any NEEs belonging to this class can be written down as

$$\begin{pmatrix} r_t(x,t) \\ q_t(x,t)\end{pmatrix}=f(\mathscr{L})\begin{pmatrix} r_x(x,t) \\ q_x(x,t)\end{pmatrix} \tag{17}$$

where f is an entire function of its arguments.

Together with the NEEs associated with the spectral operator (13), we are able to write down the corresponding M operator which gives the second linear problem

$$\varphi_{2,t}=-M\varphi_2.$$

In fact, in correspondence with a given NEE of the hierarchy characterised by a given entire function f, we can obtain the corresponding M operator by taking into account that from (16) we have

$$M^{j+1}=S(V_1^j,V_2^j,M^j,\lambda)+M^0,$$

$$S(V_1^j,V_2^j,M^j,\lambda)\varphi_2$$

$$=\lambda M^j\varphi_2+V_1^j(x,t)\varphi_2+\left(\int_x^\infty dx'\, V_1^j(x',t)\right)\varphi_{2,x}+\left(\int_x^\infty dx'\, V_2^j(x',t)\right)\varphi_2,$$

$$M^0\varphi_2=-F^0\varphi_{2,x}.$$

3.2. *The hierarchy of Bäcklund transformations*

Let us consider two linear problems of type (13) corresponding to the soliton fields $(q(x,t),r(x,t))$ and $(\tilde q(x,t),\tilde r(x,t))$: $L(q,r)\varphi_2=\lambda\varphi_2$ and $L(\tilde q,\tilde r)\tilde\varphi_2=\lambda\tilde\varphi_2$. We can now look into the general problem of finding the class of operators $\mathscr{D}^j$, W_1^j and W_2^j such that the following equation is fulfilled:

$$[L(\tilde q,\tilde r)\mathscr{D}^j-\mathscr{D}^jL(q,r)]\varphi_2=W_1^j\varphi_2+\int_x^\infty dx'\, W_2^j(x')\varphi_2(x'). \tag{18}$$

For $W_1^j = W_2^j = 0$ the operator $\mathscr{D}^j$ is a Darboux operator, i.e. such that $\tilde{\varphi}_2 = \mathscr{D}^j\varphi_2$.

The functional relation (usually nonlinear) between the soliton fields (q, r) and $(\tilde{q}, \tilde{r})$ given by $W_1^j = W_2^j = 0$ is a BT. So the problem of constructing BTs is reduced to that of the construction of a set of operators $\mathscr{D}^j$, W_1^j, W_2^j which satisfy (18). Analogously as in § 3.1 (see also Bruschi and Ragnisco 1980b) we set

$$\mathscr{D}^{j+1}\varphi_2 = \tilde{L}\mathscr{D}^j\varphi_2 + F^j\varphi_{2,x} + \int_x^\infty \mathrm{d}x'\, G^j\varphi_{2,x'} \tag{19}$$

and by requiring that (18), which is valid for $\mathscr{D}^j$, be also true for $\mathscr{D}^{j+1}$, we get

$$F^j = K(x)\Big(-F^0 + \int_x^\infty \mathrm{d}x'\, W_1^j(x')K^{-1}(x')\Big), \qquad G^j = -G^0 - \int_x^\infty \mathrm{d}x'\, W_2^j(x')$$

where $K(x) = \exp\{\int_x^\infty \mathrm{d}x'\,[\tilde{r}(x') - r(x')]\}$, G^0, F^0 are arbitrary constants independent of W_1 and W_2, and

$$\begin{pmatrix} W_1^{j+1} \\ W_2^{j+1} \end{pmatrix} = \Lambda\begin{pmatrix} W_1^j \\ W_2^j \end{pmatrix} + \begin{pmatrix} W_1^0 \\ W_2^0 \end{pmatrix} + \begin{pmatrix} \tilde{W}_1^0 \\ \tilde{W}_2^0 \end{pmatrix}$$

where

$$\Lambda\begin{pmatrix} W_1^j \\ W_2^j \end{pmatrix} = \begin{pmatrix} W_{1,x}^j + \tilde{r}W_1^j - 2W_2^j + (\tilde{r} - r)\displaystyle\int_x^\infty \mathrm{d}x'\, W_2^j(x') \\ -(\tilde{q} - q + r_x)\displaystyle\int_x^\infty \mathrm{d}x'\, W_1^j(x')K^{-1}(x') \\ -W_{2,x}^j + rW_2^j + 2\tilde{q}W_1^j + (\tilde{q} - q)\displaystyle\int_x^\infty \mathrm{d}x'\, W_2^j(x') \\ +[\tilde{q}(\tilde{r} - r) - \tilde{q}_x]K(x)\displaystyle\int_x^\infty \mathrm{d}x'\, W_1^j(x')K^{-1}(x') \end{pmatrix},$$

$$\begin{pmatrix} W_1^0 \\ W_2^0 \end{pmatrix} = G^0\begin{pmatrix} \tilde{r} - r \\ \tilde{q} - q \end{pmatrix}, \qquad \begin{pmatrix} \tilde{W}_1^0 \\ \tilde{W}_2^0 \end{pmatrix} = F^0K(x)\begin{pmatrix} \tilde{q} - q + r_x \\ \tilde{q}(\tilde{r} - r) - \tilde{q}_x \end{pmatrix}.$$

So a generic BT can be written as

$$g_1(\Lambda)\begin{pmatrix} W_1^0 \\ W_2^0 \end{pmatrix} + g_2(\Lambda)\begin{pmatrix} \tilde{W}_1^0 \\ \tilde{W}_2^0 \end{pmatrix} = 0$$

where g_1, g_2 are entire functions of their arguments and the corresponding Darboux operator can be constructed starting from (19) by taking into account that

$$\mathscr{D}^{j+1} = \textstyle\sum(W_1^j, W_2^j, \mathscr{D}^j, \lambda) + \mathscr{D}^0 + \tilde{\mathscr{D}}^0,$$

$$\begin{aligned} \textstyle\sum(W_1^j, W_2^j, \mathscr{D}^j, \lambda)\varphi_2 &= \lambda\mathscr{D}^j\varphi_2 + W_1^j\varphi_2 + K(x)\Big(\int_x^\infty \mathrm{d}x'\, W_1^j(x')K^{-1}(x')\Big)\varphi_{2,x} \\ &\quad + \Big(\int_x^\infty \mathrm{d}x'\, W_2^j(x')\Big)\varphi_2, \end{aligned}$$

$$\mathscr{D}^0\varphi_2 = -K(x)F^0\varphi_{2,x}, \qquad \tilde{\mathscr{D}}^0\varphi_2 = G^0\varphi_2.$$

3.3. *Examples of* NEEs *and* BTs

In the following we list a few of the interesting NEEs belonging to this class and the simplest non-trivial BT together with its associated Darboux operator and its matrix equivalent.

From equation (17) for $f(x)=x$ we get

$$r_t = r_{xx} + 2rr_x - 2q_x, \qquad q_t = -q_{xx} + 2qr_x + 2rq_x,$$

and for $f(x)=x^2$

$$r_t = r_{xxx} + \tfrac{3}{2}(r^2)_{xx} + (r^3)_x - 6(rq)_x, \qquad q_t = +q_{xxx} - 6qq_x - 3(q_x r)_x + 3(r^2 q)_x. \tag{20}$$

We call system (20) a CKdV system because, by the reduction $r=0$, it reduces to the KdV equation. The reduction $r=0$ can be carried out for any even function $f(x)$. In this case the operator $\mathscr{L}^2$, which transforms even elements into even elements of the hierarchy, is equal to the usual recurrence operator of the KdV hierarchy†.

By setting $q=r_x$ or $q=0$ we get, as a different reduction, the Burgers hierarchy (Levi *et al* 1983). In fact, by an appropriate substitution, the linear problem (13) gives rise to the Cole–Hopf transformation.

For the whole class of NEEs the simplest BT is obtained with the choice $g_1(x)=x$, $g_2(x)=1$. Then the Darboux operator reads

$$\mathscr{D}\varphi_2 = -[F^0K(x)+G^0]\varphi_{2,x} + G^0\left(\lambda + \tilde{r} - r + \int_x^\infty \mathrm{d}x'\,(\tilde{q}-q)\right)\varphi_2 \tag{21}$$

and the corresponding BTs are

$$F^0K(x)(\tilde{q}-q+r_x) + G^0\left(\tilde{r}_x + \tilde{r}(\tilde{r}-r) - (\tilde{q}-q) + (\tilde{r}-r)\int_x^\infty \mathrm{d}x'\,(\tilde{q}-q)\right) = 0,$$

$$F^0K(x)[\tilde{q}(\tilde{r}-r) - \tilde{q}_x] - G^0\left(q_x + \tilde{q}\tilde{r} - qr + (\tilde{q}-q)\int_x^\infty \mathrm{d}x'\,(\tilde{q}-q)\right) = 0.$$

Now by taking into account the condition (14) and linear problem (13), equation (21) can be cast in matrix form

$$D = \begin{pmatrix} G^0\int_x^\infty \mathrm{d}x'\,(\tilde{q}-q) & F^0K(x)(r_x+q) + G^0\left(\tilde{q} - \tilde{r}_x - (\tilde{r}-r)r - (\tilde{r}-r)\int_x^\infty \mathrm{d}x'\,(\tilde{q}-q)\right) \\ G^0 + F^0K(x) & F^0K(x)r + G^0\left(\tilde{r} + \int_x^\infty \mathrm{d}x'\,(\tilde{q}-q)\right) \end{pmatrix} + \lambda\begin{pmatrix} G^0 & 0 \\ 0 & -F^0K(x) \end{pmatrix}.$$

Thus the normalisation condition of the Hilbert–Riemann problem is given by

$$\chi_j = \begin{pmatrix} G^0 & 0 \\ 0 & -F^0\exp\left(\int_x^\infty \mathrm{d}x'\,[r^j(x') - r^{j-1}(x')]\right) \end{pmatrix}.$$

† After this work had been completed we took notice of the article by Iino and Ichikawa (1982) where the same spectral problem has been associated with the KdV equation.

4. Soliton solution

The CKdV hierarchy, as we have seen at the end of § 3, has non-constant normalisation condition for the Hilbert–Riemann problem. Therefore, to compute the soliton solution, we look for the gauge transformation of the linear problem (12) to the one with constant normalisation.

$D_1(q, q)$ is constant, the conditions (a) and (b) are satisfied and the gauge transformation certainly exists. Applying formula (11), we obtain

$$\bar{D} = \begin{pmatrix} G^0 & 0 \\ 0 & -F^0 \end{pmatrix}$$

and

$$A(q) = \begin{pmatrix} 1/G^0 & 0 \\ 0 & -\frac{1}{F^0}\exp\left(-\int_x^\infty dx'\, r(x')\right) \end{pmatrix}.$$

From (10) the transformed linear problem takes the form

$$\psi_x = \begin{pmatrix} 0 & u \\ v & \lambda \end{pmatrix}\psi \tag{22}$$

where $u = (F^0/G^0)q\exp(\int_x^\infty dx'\, r(x'))$ and $v = (G^0/F^0)\exp(-\int_x^\infty dx'\, r(x'))$. For simplicity of exposition we apply here the formulae (7) and (8) only to the case of $n = 1$. This solution, though being the simplest one, still shows some interesting properties. Then, starting from the 'bare' solution $q = r = 0$, the one-soliton solution of the linear problem (22) reads

$$\begin{pmatrix} 0 & u \\ v+1 & 0 \end{pmatrix} = (\lambda_1 - \mu_1)\left[P_1, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

where P_1 is a projection matrix, which can be constructed in the usual way (Levi *et al* 1980). The explicit formula for the one-soliton fields has the form

$$\begin{aligned} q(x,t) &= (\lambda_1 - \mu_1)\lambda_1\mu_1 \frac{[\lambda_1 \exp\chi_1(1+\exp\chi_2) - \mu_1\exp\chi_2(1+\exp\chi_1)]}{[\lambda_1(1+\exp\chi_2) - \mu_1(1+\exp\chi_1)]^2}, \\ r(x,t) &= (\lambda_1 - \mu_1)[\lambda_1^2\exp\chi_1(1+\exp\chi_2)^2 - \mu_1^2\exp\chi_2(1+\exp\chi_1)^2] \\ &\quad\times[\lambda_1\exp\chi_1(1+\exp\chi_2) - \mu_1\exp\chi_2(1+\exp\chi_1)]^{-1} \\ &\quad\times[\lambda_1(1+\exp\chi_2) - \mu_1(1+\exp\chi_1)]^{-1} \end{aligned} \tag{23}$$

where $\chi_1 = \lambda_1(x - f(\lambda_1)t - x_{10})$, $\chi_2 = \mu_1(x - f(\mu_1)t - x_{20})$ and the function f, introduced in (17), defines the particular system for which (23) is the one-soliton solution, while x_{10} are arbitrary real constants. Moreover, the requirement that q and r should be real, everywhere finite, fields with zero asymptotic values implies that λ_1 and μ_1 are to be real and of opposite signs.

In figure 1 we plot the evolution of the one-soliton solution (23) for the CKdv system (20). From figure 1 we can see that

(a) the q-field peaks maintain the KdV one-soliton shape; however, the 'one-soliton' peak coming from $-\infty$ splits into two 'one-soliton' peaks†;

† Both q and r fields conserve their area during the evolution.

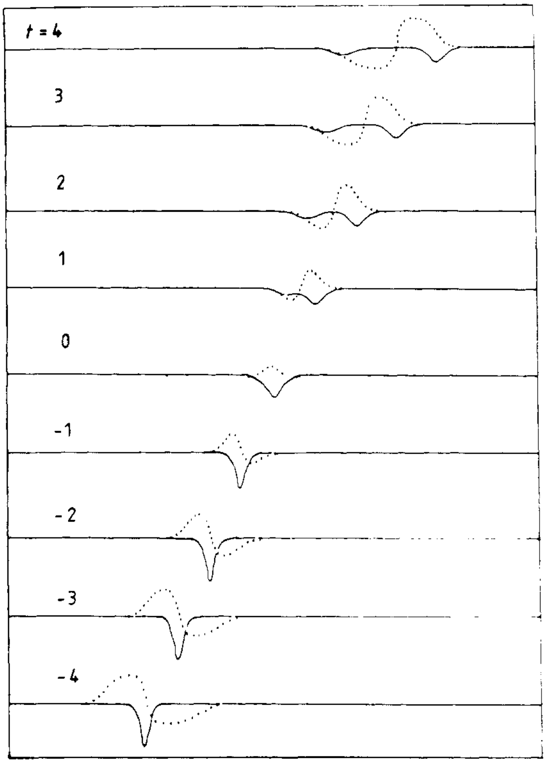

Figure 1. The one-soliton solution of CKdV (20) for $\lambda_1 = 2$, $\mu_1 = -1.4$, $x_{10} = x_{20} = 0$; q is depicted by a continuous line, while r is dotted.

(b) the r field is built up from three kink-like objects; the evolution transforms the past configuration, two 'kink' + one 'antikink', to the configuration one 'kink' + two 'antikink'.

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